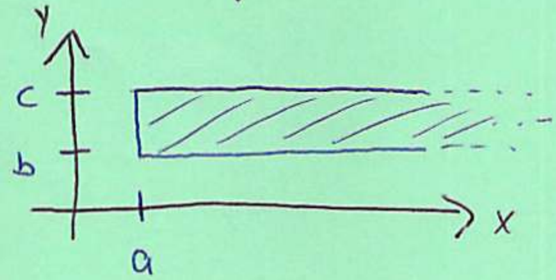
A set is <u>bounded</u> in $\mathbb{R}^2$ if it can fit in a disk. All of the above examples are bounded. An unbounded example is:



$$\{(x,y) \mid x \geq a,\ b \leq y \leq c\}$$

Lecture 11

Procedure for finding extrema on closed & bounded sets

This is the analog to finding extrema of $y=f(x)$ on $[a,b]$.

To find the absolute maximum and minimum values of a continuous function $f=f(x,y)$ on a closed & bounded set D:

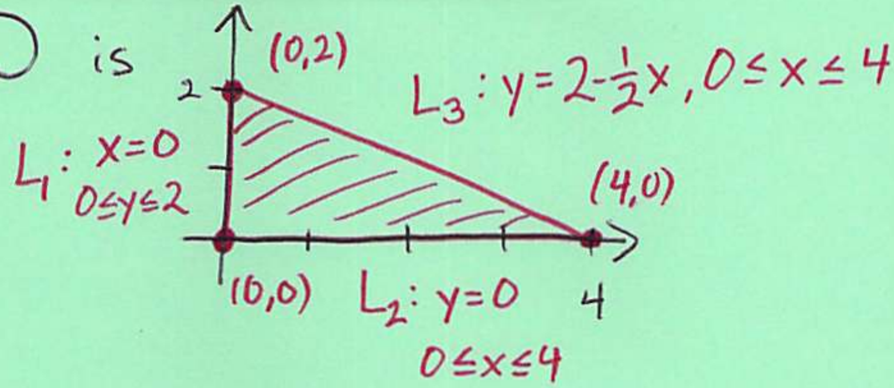
① Find the values of f @ critical points of f <u>inside</u> D.

② Find the extreme values of f on the boundary.

③ The largest value from ① & ② is the absolute max & the smallest is the absolute min.

<u>Ex</u>: Find the absolute maximum & minimum values of $f(x,y)=x+y-xy$ on the closed & bounded set D which is the closed triangle with vertices $(0,0)$, $(0,2)$, and $(4,0)$.

Sol: The region D is



$\nabla f = \langle 1-y, 1-x\rangle$, so the critical point of f is $(1,1)$, which is inside D.

Now, we check on the boundary of D:

L_1: Plug $x=0$ into f to restrict to L_1:

$$g_1(y) = f(0,y) = y.$$

Now, treat g_1 as defined on $[0,2]$, and do calc I: we must include the endpoints $y=0,2$ as possible extrema. Since $g_1'(y)=1$, there are no extrema inside $[0,2]$. So, L_1 gives us possible extrema at $(0,0)$ & $(0,2)$.

L_2: Plug in $y=0$: $g_2(x)=f(x,0)=x$, $0\le x\le 4$.
$g_2'(x)=1$, so we only get the endpoints as possibilities: $(0,0)$ & $(4,0)$.

L_3: Plug in $y=2-\frac{1}{2}x$: $g_3(x)=f(x,2-\frac{1}{2}x)=x+(2-\frac{1}{2}x)-x(2-\frac{1}{2}x)$

$$= x+2-\tfrac{1}{2}x-2x+\tfrac{1}{2}x^2 = \tfrac{1}{2}x^2-\tfrac{3}{2}x+2$$

$0\le x\le 4$.

$g_3'(x) = x - \frac{3}{2}$. $g_3'(x) = 0$ when $x = \frac{3}{2}$.

So, there is a potential extremum at $\left(\frac{3}{2}, 2 - \frac{1}{2}\left(\frac{3}{2}\right)\right) = \left(\frac{3}{2}, \frac{5}{4}\right)$, as well as the endpoints $(0,2)$ & $(4,0)$.

So, all the potential points & their values are:

Point	Value of f	
$(0,0)$	0	$\leftarrow$ abs min
$(4,0)$	4	$\leftarrow$ abs max
$(0,2)$	2	
$\left(\frac{3}{2}, \frac{5}{4}\right)$	$\frac{3}{2} + \frac{5}{4} - \left(\frac{3}{2}\right)\left(\frac{5}{4}\right) = \frac{7}{8}$	
$(1,1)$	$1+1-1 = 1$	

◇

Ex: A cardboard box without a lid is to have a volume of 32000 cm^3. Find the dimensions of the box which uses the least amount of cardboard.

Sol: The Volume is constant: $V = lwh = 32000$

We aim to minimize surface area:

$$A = \underset{\uparrow}{lw} + 2(hw + hl)$$

not multiplied by 2 since there is no lid.

We only have tools to work w/ 2 variables, so we need to eliminate one, say h: Using the volume, we find $h = \frac{32000}{lw}$ (One should be careful if l or $w=0$, but those are not acceptable solutions!)

$$\Rightarrow A = lw + 2\left(\frac{32000}{l} + \frac{32000}{w}\right) = lw + 64000\left(\frac{1}{l} + \frac{1}{w}\right)$$

$$\nabla A = \left\langle \frac{\partial A}{\partial l}, \frac{\partial A}{\partial w} \right\rangle = \left\langle w - \frac{64000}{l^2}, l - \frac{64000}{w^2} \right\rangle$$

$$\nabla A = \vec{0} \Rightarrow \begin{cases} w - \frac{64000}{l^2} = 0 & (1) \\ l - \frac{64000}{w^2} = 0 & (2) \end{cases}$$

(1) $\Rightarrow w = \frac{64000}{l^2}$. Plug into (2):

$$l - 64000\frac{1}{w^2} = l - 64000\left(\frac{l^4}{(64000)^2}\right) = l - \frac{l^4}{64000}$$

$$= l\left(1 - \frac{l^3}{64000}\right) = 0$$

Since we don't want $l=0$, $l=40$ is the only solution.

When $l=40$, $w = \frac{64000}{40^2} = \frac{40^3}{40^2} = 40$.

Now, we make sure it's a minimum:

$A_{ll} = \dfrac{128000}{l^3}$, $A_{ll}(40,40) = \dfrac{128000}{40^3} = \dfrac{2 \cdot 40^3}{40^3} = 2 > 0$

$$HA = \begin{pmatrix} \frac{128000}{l^3} & 1 \\ 1 & \frac{128000}{w^3} \end{pmatrix}, \quad HA(40,40) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$D(40,40) = 4 - 1 = 3 > 0$. So $(40,40)$ is a min.

If $l = w = 40$, $V = lwh = 32000 \Rightarrow h = 20$, so the dimensions are $40\text{cm} \times 40\text{cm} \times 20\text{cm}$. ◇

14.8 - Lagrange Multipliers

Suppose we want to extremize $f(x,y) = x^2 + 2y^2$ subject to the constraint $x^2 + y^2 = 1$. (this is restricting the domain of f to the constraint and finding max & min values). We'll approach this geometrically (see mathematica code). We can see that the smallest & largest valued level curves intersect the graph of the constraint tangently, that is, if we view the constraint as a level curve of a function: $g(x,y) = k$, ($g(x,y) = -x^2 - y^2 = -1$ in this example), then $\nabla f \parallel \nabla g$. Let's formalize this (in 3 variables, but 2 works as well):

Say we want to extremize $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$, a level surface of g. Suppose $P=(x_0,y_0,z_0)$ is an extreme point and let $\vec{r}(t)=\langle x(t),y(t),z(t)\rangle$ be a curve on the constraint surface such that $\vec{r}(t_0)=P$. We've learned that $\nabla g(P) \perp \vec{r}\,'(t_0)$. Recall that we are looking for the "highest and lowest" points on the graph of $f(x,y,z)$ over the surface $g=k$. So, since $\vec{r}(t)$ lies on $g=k$, we can plug it into f and get: $h(t)=f(\vec{r}(t))$, the values of f over $\vec{r}$. Since $\vec{r}(t_0)=P$, $h(t)$ hits an extremum at t_0, meaning that $h'(t_0)=0$. So:

$$h'(t_0)=\nabla f(\vec{r}(t_0))\cdot \vec{r}\,'(t_0)=0$$

But, this is true for all curves $\vec{r}(t)$ in $g=k$ passing through P. So, we have $\nabla f \perp g=k$ at P. But, since $\nabla g \perp g=k$ at P, that must mean $\nabla f(P) \parallel \nabla g(P)$. So, $\nabla f(P)=\lambda \nabla g(P)$.

λ is called a <u>Lagrange Multiplier</u>.

Method of Lagrange Multipliers :

Assume that $\nabla g \neq \vec{0}$ on $g=k$ and that extreme values of $f=f(x,y,z)$ subject to $g=k$ exist. To find them:

ⓐ Find all quadruples (x,y,z,λ) solving the system:

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = k \end{cases} \iff \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = k \end{cases}$$

ⓑ Evaluate f at all points (x,y,z) from ⓐ and identify the maxima/minima.

Ex: Find the extreme values of $f(x,y) = x^2+2y^2$ subject to the constraint $x^2+y^2=1$.

Sol: $g=x^2+y^2$. $\nabla f = \langle 2x, 4y\rangle$, $\nabla g = \langle 2x, 2y\rangle$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g=1 \end{cases} \Rightarrow \begin{cases} 2x = \lambda 2x & ① \\ 4y = \lambda 2y & ② \\ x^2+y^2 = 1 & ③ \end{cases}$$

① $\Rightarrow x=0$ or $\lambda=1$

$\boxed{x=0}$ ③ $\Rightarrow y^2=1 \Rightarrow y=\pm 1$. $y=1 \overset{②}{\Rightarrow} \lambda=2$

$y=-1 \Rightarrow \lambda=-2$

(Note: We don't actually need the values of λ, we just need to know they can be found.)

Candidate points: $(0,1)$ & $(0,-1)$.

$\boxed{\lambda = 1}$ ② $\Rightarrow y = 0 \overset{③}{\Rightarrow} x^2 = 1 \Rightarrow x = \pm 1$

Candidate points : $(1,0)$ & $(-1,0)$.

Now, we check :

Point	Value of f	
$(0,1)$	$f(0,1) = 2$	max
$(0,-1)$	$f(0,-1) = 2$	
$(1,0)$	$f(1,0) = 1$	min
$(-1,0)$	$f(-1,0) = 1$	

$\diamond$

Let's see an example which also brings in ideas from the previous section.

Ex: Find the extreme values of $f(x,y) = e^{-xy}$ on the region $x^2 + 4y^2 \leq 1$.

Sol: Lagrange multipliers will only find extreme values of $f = f(x,y)$ when restricted to a level curve $g = k$. So, Lagrange multipliers will only work on the boundary of this region.

Let's first do the interior:

$\nabla f = \langle -ye^{-xy}, -xe^{-xy} \rangle$. Since $e^{-xy} > 0$ for all (x,y), it follows that $\nabla f = \vec{0}$ only at $(0,0)$.

Now, we check the boundary:

The boundary is $x^2+4y^2=1$, so we can see it as a level curve of $g(x,y)=x^2+4y^2$: $g=1$.

$\nabla g = \langle 2x, 8y \rangle$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g=1 \end{cases} \Rightarrow \begin{cases} -ye^{-xy} = \lambda 2x & ① \\ -xe^{-xy} = \lambda 8y & ② \\ x^2+4y^2 = 1 & ③ \end{cases}$$

Let's try solving for e^{-xy}. Using ①, we need to check what happens if $y=0$.

$\boxed{y=0}$ ① $\Rightarrow \lambda=0$ or $x=0$

$x=0$ contradicts ③

$\lambda=0$, by ② $\overset{(e^{-xy}>0)}{\Rightarrow} x=0$, which again contradicts ③

So, we have that $y=0$ cannot happen, so $y \neq 0$.

Playing this same game, this time starting with ②, we find that $x=0$ cannot happen either. So, $x \neq 0$. Now, ① & ② give

$$\frac{2\lambda x}{y} \overset{①}{=} -e^{-xy} \overset{②}{=} \frac{8\lambda y}{x} \iff \frac{2\lambda x}{y} = \frac{8\lambda y}{x}.$$ Multiply through by xy:

$2\lambda x^2 = 8\lambda y^2 \iff \lambda x^2 = 4\lambda y^2$. Now, by ③ $x^2 = 1-4y^2$, so plug this in: $\lambda(1-4y^2) = 4\lambda y^2 \Rightarrow \lambda = 8\lambda y^2$. So, $\lambda=0$ or $8y^2=1$.

$\boxed{\lambda=0}$ ① $\Rightarrow y=0$, which we know doesn't happen. So $\lambda \neq 0$.

$\Rightarrow 8y^2=1 \Rightarrow y^2=\frac{1}{8} \Rightarrow y=\pm\frac{1}{2\sqrt{2}}$. By ③: $x^2 = 1-4y^2 = 1-4\left(\frac{1}{8}\right) = 1-\frac{1}{2} = \frac{1}{2}$

$\Rightarrow x = \pm\frac{1}{\sqrt{2}}$

So the candidate points and the values of f at them are:

Candidate	Value of f	
$(0,0)$	1	
$\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$	$e^{-\frac{1}{4}}$	min
$\left(\frac{1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right)$	$e^{1/4}$	max
$\left(\frac{-1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$	$e^{1/4}$	max
$\left(\frac{-1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right)$	$e^{-1/4}$	min

$e^{1/4} > 1$ since $e > 1$

$\Rightarrow e^{-1/4} < 1$

$\diamond$

Lecture 12

Two Constraints: Suppose we want to extremize $f(x,y,z)$ subject to two constraints: $g(x,y,z)=c$ & $h(x,y,z)=k$. Geometrically, we are extremizing f along the curve of intersection of $g=c$ & $h=k$. Now, we still have that ∇f is perpendicular to the curve of intersection at an extreme point, P, but it isn't necessarily perpendicular to both $g=c$ & $h=k$ at this point. However, since both ∇g & ∇h are perpendicular to the curve at this point, we find that $\nabla f(P) = \lambda \nabla g(P) + \mu \nabla h(P)$. To summarize, we now have to solve

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) = c \\ h(x,y,z) = k \end{cases}$$